It is best to use the above formula when $b \gg 1$, otherwise we have to apply formula 9.131 and 9.132 from /5/. In order to demonstrate their use, we will obtain the NSIC for the case when $b = 0$, i.e. for the case when the force P is applied directly to the edge $x = +0$ of the crack. In this case Gauss's function in (5.10) should be transformed using formula 9.132(2) from /5/, and this will enable us to carry out, after some reduction, the passage to the limit as $b \to 0$. As a result we obtain

$$N_1^{(0)} = \pi^{-2} P \, (2a)^{-3/2} (8 - 3\mu) (1 - \mu)^{-1}.$$

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FRACTURE OF A NARROW BRIDGE BETWEEN CRACKS LYING IN THE SAME PLANE*

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The stress-deformation state of an isotropic elastic space weakened by a family of cracks of normal separation is investigated. In some regions the crack edges come closer to each other, and form narrow bridges (ligaments). An asymptotic form of the solution of the problem is constructed under the assumption that the bridge either contracts to a contour, or becomes an open arc. Special features of the stresses at the tip of the bridge are studied for various forms of the tip. Asymptotic formulas obtained are used to produce variational formulations of the problems, and the lack of uniqueness of these solutions is interpreted as the instability of the process of disruption of narrow bridges. Examples are considered.

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1. A bridge contracting to a closed contour. Let G denote a region in the plane $\mathbf{R}^2$ with a smooth (class C^∞) boundary ∂G, where G is either a bounded set or the outside of a bounded set. In addition, let Γ be an arbitrary smooth closed contour of length l, situated in G, and let (n, s) be a natural system of local coordinates in the neighbourhood of Γ ($|n|$ is the distance along the normal and s is the arc length). We put $\Gamma_\varepsilon = \{(n, s): s \in [0, l), -\varepsilon h_-(s) \leqslant n \leqslant \varepsilon h_+(s)\}$ and $G_\varepsilon = G \setminus \Gamma$, and here $h_\pm$ are smooth positive functions, $h = h_+ + h_-$ is the width of the bridge and $0 < \varepsilon$ is a small parameter (all coordinates are dimenionless). We shall consider a homogeneous elastic isotropic space $\mathbf{R}^3$ with a crack G_ε in the plane $\{x = (y, z): z = 0\}$, and a symmetric normal load $p(y)$ which is a function which is smooth in $\bar{G}$, is applied to the edges $G_\varepsilon^\pm$ of the crack. Using the Papkovich-Neuber representation we can reduce problem /1/ to that of determining the function v^ε harmonic in the half-space $\mathbf{R}_+^3 = \{x : z > 0\}$, and satisfying the conditions

$$v^\varepsilon(y, 0) = 0, \quad y \in \mathbf{R}^2 \setminus G_\varepsilon; \quad (\partial_z v^\varepsilon)(y, 0) = P(y) \equiv -\mu^{-1}(1-\nu)p(y), \quad y \in G^\varepsilon \tag{1.1}$$

Here ν is Poisson's ratio, μ is the shear modulus, $\partial_z = \partial/\partial z$. When $\varepsilon \to 0$, the bridge Γ_ε contracts to a closed contour Γ. The asymptotic form of solutions of the problem in the regions with similar singular perturbation of the boundary was studied in /2-5/. The principal term of the expansion of v^ε is sought, as $\varepsilon \to 0$, in the form

$$v^0(x) = -\int_G P(\eta)\Phi(x; \eta, 0)\,d\eta - \int_\Gamma \gamma(\tau)\Phi(x; \eta, 0)\,d\tau \tag{1.2}$$

In (1.2), Φ is Green's function for Poisson's equation in $\mathbf{R}_+^3$ with boundary conditions of the type (1.1), and τ is the coordinate of the point $\eta = (\eta_1, \eta_2)$ on the arc Γ. The first integral corresponds to the load p, and the second integral corresponds to the normal force distributed along the contour Γ, at some smooth density $\mu(1-\nu)^{-1}\gamma$. In order to obtain the equation for γ, we shall find the asymptotic expansion for the function (1.2) at $r = (n^2 + z^2)^{1/2} \to 0$. Since $\Phi(x; \eta, 0) = (2\pi)^{-1}(|y - \eta|^2 + z^2)^{-1/2} + O(1)$, it follows that the regularization of the second integral in (1.2) leads to the formulas

$$v^0(x) = \gamma(s)(\pi^{-1}\ln(1/_8 k(s) r) - b(s)) + (J\gamma)(s) - g(s) + O(r|\ln r|) \tag{1.3}$$
$$(r \to 0)$$

$$b(s) = \int_\Gamma \{\Phi(y, 0; \eta, 0) - \chi(\tau - s)\,\Xi(\tau, s)\}\,d\tau - \int_{|\tau| < \pi/k(s)} (1 - \chi(\tau - s))\,\Xi(\tau, s))\,d\tau \tag{1.4}$$

$$\Xi(\tau, s) = (2\pi)^{-1}k(s)\,[2(1 - \cos[(\tau - s)k(s)])]^{-1/2}$$

$$g(s) = \int_G P(\eta)\Phi(y, 0; \eta, 0)\,d\eta$$

$$(J\gamma)(s) = \int_\Gamma (\gamma(s) - \gamma(\tau))\Phi(y, 0; \eta, 0)\,d\tau \tag{1.5}$$

Here $\chi \in C_0^\infty(-d, d)$ is the cutoff function, $\chi(\tau) = 1$ when $|\tau| \leqslant 1/_2 d$, $0 < d$ is a small number, and $k(s)$ is the curvature of the contour Γ at the point s. We note that the factor accompanying γ in (1.3) has no singularities when $k(s^\circ) = 0$, and according to (1.4) the difference $\ln k(s) - \pi b(s)$ remains bounded when $s \to s^\circ$. It can be confirmed that the integral operator (1.5) is a pseudodifferential operator with the principal symbol $\pi^{-1}\ln|\xi|$.

A boundary layer appears near the bridge Γ_ε. We shall now introduce the "stretched" variables $\zeta = (\zeta_1, \zeta_2) = [\varepsilon h(s)]^{-1}(n, z)$. Let us change to the coordinates ζ, s and write $\varepsilon = 0$. As a result we obtain the following parametric problem depending on $s \in \Gamma$ for the function $w^0(\zeta, s)$ of the boundary layer type:

$$\Delta_\zeta w^0(\zeta, s) = 0, \quad \zeta \in \mathbf{R}_+^2 = \{\zeta : \zeta_2 > 0\}$$
$$(\partial w^0/\partial \zeta_2)(\zeta_1, 0, s) = 0, \quad \zeta_1 \overline{\in} (0, 1), \quad w^0(\zeta_1, 0, s) = 0, \quad \zeta_1 \in (0, 1)$$

The solution of this problem, which increases logarithmically at infinity, is given by the equation

$$w^0(\zeta, s) = a(s)\ln|2\zeta - h^{-1}(h_+ - h_-) + 2[(\zeta - h_+h^{-1})(\zeta + h_-h^{-1})]^{1/2}|, \tag{1.6}$$
$$\zeta = \zeta_1 + i\zeta_2$$

Using the conditions of matching the expansions (1.3) with the expansion of $w^0(\zeta, s) = a(s)\ln(4|\zeta|) + O(|\zeta|^{-1})$, $|\zeta| \to \infty$, we obtain the relations

$$a(s) = \pi^{-1}\gamma(s) \tag{1.7}$$

$$[\pi^{-1}(\ln(\varepsilon h(s)\, k(s)) - 5\ln 2) - b(s)]\,\gamma(s) + (J\gamma)(s) = g(s) \tag{1.8}$$

Thus the density γ represents the solution of integral Eq.(1.8). Since $\Phi(y, 0; \eta, 0)$ is a positive symmetric function of the variables y and η, it follows that J is a symmetric and non-negative definite operator. In particular, we have

$$\int_\Gamma \gamma(s)(J\gamma)(s)\,ds = \frac{1}{2}\int_\Gamma\int_\Gamma |\gamma(s) - \gamma(\tau)|^2\,\Phi(y, 0; \eta, 0)\,ds d\tau \tag{1.9}$$

Let us now denote by $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and $\psi_1, \psi_2, \ldots$ the sequences of the eigenvalues and corresponding eigenfunctions of the operator

$$I_h = \pi^{-1}(\ln(hk) - 5\ln 2) - b + J \tag{1.10}$$

normed in $L_2(\Gamma)$. We note that when $\varepsilon = \varepsilon_k \equiv \exp(-\pi\lambda_k)$, Eq.(1.8) has, generally speaking, no solution, and $\varepsilon_k \to 0$ as $k \to \infty$. It follows therefore that we have to determine, just as in /2-5/, only the asymptotic solution. Let us write

$$g(s) = \sum_{k=1}^{\infty} g_k\psi_k(s), \quad \gamma_\varepsilon(s) = \sum_{k \leqslant \varepsilon^{-1/2}} (\lambda_k + \pi^{-1}\ln\varepsilon)^{-1} g_k\psi_k(s) \tag{1.11}$$

Since g is a smooth function on Γ, it follows that γ_ε satisfies Eq.(1.8) with the accuracy of $O(\varepsilon^N)$ for large N.

Thus we have constructed the smooth-type function (1.3) and boundary layer-type function (1.6). Matching these functions yields a global asymptotic approximation to the solution of Eq.(1.1). The estimate $O(\varepsilon\,|\ln\varepsilon|)$ of the residue in the asymptotic expression is checked just as in /3, 5/.

The relation $|v(x) - w^0(\zeta, s)| \leqslant c\varepsilon\,|\ln\varepsilon|$ holds near the bridge Γ_ε. This relation, together with (1.6) and (1.7), leads to the following formula for the stress intensity coefficient (SIC) at the edges $\Gamma_\varepsilon^\pm = \{x : z = 0,\ s \in [0, l),\ n = \pm\varepsilon h(s)\}$ of the cracks:

$$v(x) = \mu^{-1}(1 - \nu)(2r_\pm\pi^{-1})^{1/2} K_\pm(s, \varepsilon)\sin{}^1/{}_2\varphi_\pm \tag{1.12}$$

$$K_\pm(s, \varepsilon) = (\pi\varepsilon h(s)/2)^{-1/2}\mu(1 - \nu)^{-1}\gamma(s) + O(\varepsilon^{1/2}\,|\ln\varepsilon|)$$

Here $(r_\pm, \varphi_\pm)$ are polar coordinates in the planes perpendicular to the contours $\Gamma_\varepsilon^\pm$.

Eq.(1.8) contains a large parameter $|\ln\varepsilon|$, and the solution γ_ε can, in turn, be expanded in a series in inverse powers of $|\ln\varepsilon|$

$$\gamma_\varepsilon(s) \sim \sum_{k=0}^{\infty} |\ln\varepsilon|^{-k-1}\gamma^{(k)}(s) \tag{1.13}$$

$$\gamma^{(0)}(s) = -\pi g(s), \quad \gamma^{(k+1)}(s) = \pi(I_h\gamma^{(k)})(s)$$

We note that when $g < 0$ on Γ and for small $\varepsilon > 0$ the density γ_ε is positive i.e. we have an extension taking place around the bridge.

Let $G = \mathbf{R}^2$ and $\Gamma_\varepsilon = \{y : R < |y| < R + \varepsilon\}$ (two half-spaces joined along a thin ring). In this case $k(s) = R^{-1}, h(s) = 1, b(s) = 0$, and

$$(J\gamma)(s) = \frac{1}{4\pi}\int_0^{2\pi} (\gamma(s) - \gamma(\tau))\left|\sin\frac{\mathrm{T} - s}{2R}\right|^{-1} d\tau$$

We shall discuss two versions of the loading: 1°. Normal forces q applied at the point $y = 0$; 2°. A uniform normal load of constant intensity p° at the edges of internal circular crack. By virtue of the axial symmetry the densities γ_1 and γ_2 are constant, and the integral equation becomes algebraic. The solutions are represented by $\gamma_1 = 4\pi\mu^{-1}(1 - \nu)R\omega q$ and $\gamma_2 = (2\mu)^{-1}(1 - \nu)\omega p^\circ$, respectively, where $\omega = (\ln(\varepsilon R^{-1}) - 5\ln 2)^{-1}\pi$. Formulas (1.12) supply the asymptotic form to the SIC with an accuracy of the order of $O(\varepsilon^{1/2}|\ln\varepsilon|)$.

$$K_\pm^{(1)}(s, \varepsilon) \sim (\pi\varepsilon/2)^{-1/2} 4\omega Rq, \quad K_\pm^{(2)}(s, \varepsilon) \sim (2\pi\varepsilon)^{-1/2}\omega p^\circ(\pi R)^{-1}$$

2. ***A bridge contracting to an arc.*** We shall use the same notation as in Sect.1, and denote by $\Gamma' \subset \Gamma$ the arc of length l, terminating at P_0 and P_l. We shall assume that the set Γ_ε has the form $\{(n, s) : s \in [0, l],\ -\varepsilon h_-(s) \leqslant n \leqslant \varepsilon h_+(s)\}$. First we shall discuss the possible form of the bridge near the point P_0 (or P_l). We shall consider the following four versions: $1^\circ.\ h_\pm(0) = 0, h_\pm'(0) \neq 0$ (angle), $2^\circ.\ h_\pm(0) = h_\pm'(0) = 0$ (peak), $3^\circ.\ h_\pm(s) = s^{1/2}(p_0 +$

$O(s)$) and $s \to +0$ (small but positive radius of curvature of the tip of the bridge Γ_ε); 4°. $h_\pm(0) \neq 0$ (a blunt tip).

1°. The region $\mathbf{R}^3 \setminus \Gamma_\varepsilon$ is diffeomorphic near the point P_0 with a cone produced by removing from the space a crack, angular in the plane, with an aperture angle of the order of $\varepsilon h'(0)$ (Dirichlet conditions are prescribed at the crack edges). The stress-deformation state near a similar singularity is determined by the stress singularity factor $\Lambda(\varepsilon)$. The factor represents an eigenvalue of some spectral problem in the region cut out by the cone from the unit sphere. The asymptotic expansion of such eigennumbers has the form /6/ $\Lambda(\varepsilon) = -1-(2\ln\varepsilon)^{-1} + O(|\ln\varepsilon|^{-2})$. Since the SIC on Γ_ε behaves like $s^{\Lambda(\varepsilon)+1/2}$ as $s \to +0$, it follows that the crack is not in the state of local equilibrium near P_0.

2°. Let us calculate the asymptotic expansion of the stress-deformation state near the tip of a plane crack, representing the outside of the peak $S = \{y \in \mathbf{R}^2 : y_1 > 0,\ |y_2| \leqslant m y_1^n\}$ where $m > 0$ and $n = 2, 3, \ldots$. We will denote by ρ, θ, φ the spherical coordinates with centre at the point O (the north pole of the unit sphere is placed at the point $(1, 0, 0)$), and the variables $\xi_j = m^{-1}x_1^{-n}x_{j+1},\ j = 1, 2$ are denoted by $\xi = (\xi_1, \xi_2)$. Let us consider the function V harmonic in the neighbourhood of O, subjected to the homogeneous Dirichlet conditions on S. According to /7-9/ the asymptotic form of the function V as $\rho \to 0$ is sought in the form

$$V(x) = C\,|\ln\rho|^{-\beta}\{\chi(\theta^{1/2})(1+\alpha\,|\ln\rho|^{-1}\ln(\sin{}^1\!/_2\theta)) + (1-\chi(\theta^{1/2}))\,\alpha\,|\ln\rho|^{-1}\,w(\xi_1, \xi_2)$$

Here χ is the same cutoff as in (1.5), w is the function (1.6) at $a(s) = 1$, C is a constant depending on V and α and β are constants to be determined.

Matching the asymptotic forms of the first two terms within the curly brackets as $\theta \to 0$ with the asymptotic forms of the third term as $|\xi| \to \infty$, we find that $\alpha = (n-1)^{-1}$. Further, when $\theta > \text{const}$, we have

$$\Delta V(x) = C\rho^{-2}\,|\ln\rho|^{-\beta-1}\{\beta - {}^1\!/_2\alpha + O(|\ln\rho|^{-1})\}\quad (\rho \to 0)$$

Therefore $\beta = [2(n-1)]^{-1}$, and this means that outside of a small conical neighbourhood of the point O the stresses are of the order of $\rho^{-1}\,|\ln\rho|^{-\beta-1}$ and the SIC at the crack edge is of th order of $O(\rho^{-n/2}\,|\ln\rho|^{-\beta-1})$. It is clear that such a crack cannot be in equilibrium.

3°. Near the point P_0 the edge of the bridge is smooth, and unlike in the previous case, the SIC remains bounded when $s \to 0$. The corresponding boundary layer is constructed at the tip of this boundary, and it is there that case 4° will now be discussed.

Let us consider problem (1.1) in assumption 3°. The principal term of the asymptotic expression for its solution is given, as before, by formula (1.2) in which the contour Γ has been replaced by the arc Γ'. The asymptotic expression (1.3) is also retained, with b and J and $O(r\,|\ln r|)$ replaced by b', J' and $O(\delta(x)\,|\ln\delta(x)|)$ respectively. Here $\delta(x) = r\,[s(l-s)]^{-1}$, J' is the integral operator (1.5) on the arc Γ', and

$$\begin{gathered} b'(s) = b(s) + \int\limits_{\Delta_-(s)} [\chi(\tau-s)\,\Xi(\tau-s) - (2\pi)^{-1}(|\tau-s|^{-1} - |s-d|^{-1}\ln s/d)]\,d\tau + \int\limits_{\Delta_+(s)} \Big[\chi(\tau-s)\,\Xi(\tau,s) - (2\pi)^{-1}\Big(|\tau-s|^{-1} - \\ |s+d-l|^{-1}\ln\frac{l-s}{d}\Big)\Big]\,d\tau \\ (\Delta_-(s) = (s-d,\ 0),\ \Delta_+(s) = (l,\ s+d)) \end{gathered} \tag{2.1}$$

In (2.1) b is given by (1.4). We assume that $(\alpha, \beta) = \varnothing$ when $\alpha \geqslant \beta$. Just as in Sect.1, the boundary layer (1.6) leads to Eq.(1.7) and an integral equation on the arc Γ' analogous to (1.8)

$$[\pi^{-1}(\ln(\varepsilon h(s)\,k(s)) - 5\ln 2) - b'(s)]\,\gamma(s) + (J'(\gamma))(s) = g(s) \tag{2.2}$$

We note that according to (2.1) and assumption 3° concerning the form of the bridge near the points P_0 and P_l, the following relations hold:

$$\begin{gathered} b'(s) = (2\pi)^{-1}\ln s + O(1),\ h(s) = s^{1/2}(2p_0 + o(1))\quad (s \to +0) \\ b'(s) = (2\pi)^{-1}\ln|l-s| + O(1),\ h(s) = |l-s|^{1/2}(2p_l + o(1)) \\ (s \to l-0) \end{gathered} \tag{2.3}$$

Therefore, the expression within the square brackets in (2.2) is bounded in the interval $[0, l]$ and the asymptotic solution γ_ε of Eq.(2.2) will be given by the formulas (1.11) with obvious transformations. However, unlike in the case of a closed contour the second integral of (1.2) will have logarithmic singularities near the ends P_0 and P_l of the arc Γ'. This leads to the need to construct a new type of boundary-layer.

Let us introduce the "stretched" three-dimensional coordinates ξ where $\xi_1 = \varepsilon^{-2}s$, $\xi_2 = \varepsilon^{-2}n$, $\xi_3 = \varepsilon^{-2}z$. We change, in the neighbourhood of the point P_0, to the variables ξ, and put $\varepsilon = 0$. As a result the set G_ε will be transformed into a parabola $\Pi = \{\xi \in \mathbf{R}^3 : \xi_1 > 0, |\xi_2| \leqslant p_0\xi_1^{1/2}, \xi_3 = 0\}$ (see the representation in (2.3)). The solution $z^0(\xi)$ of the three-dimensional boundary-layer type will now be a function that is harmonic in $\mathbf{R}_+^3$ and satisfying the following boundary conditions:

$$z^0(\xi) = 0, \ \xi \in \Pi; \ (\partial z^0/\partial\xi_3)(\xi) = 0, \ \xi \in \partial\mathbf{R}_+^3 \setminus \overline{\Pi} \tag{2.4}$$

In order to find the conditions which should be imposed on the function z^0 at infinity, we note that

$$\int_{\Gamma'} \gamma(\tau)\,\Phi(x;\eta,0)\,d\tau = -(2\pi)^{-1}\gamma(0)\ln[(r^2+s^2)^{1/2} - s] + O(1)$$

Thus the conditions of matching yield the representation

$$z^0(\xi) = \gamma(0)(2\pi)^{-1}\ln(|\xi|^{1/2} - \xi_1) + O(1) \quad (|\xi| \to \infty) \tag{2.5}$$

We stress that the above representation holds outside the conical neighbourhood of the ray $\{\xi : \xi_1 > 0, \xi_2 = \xi_3 = 0\}$, and a two-dimensional boundary layer analogous to (1.6) appears near the parabola.

The solution required is derived from the well-known solution of the problem of an external elliptical crack /10/. Indeed, let the ellipse have the semiaxes 1 and $2^{-1/2}p_0\varepsilon$. After the change of variables $\xi_1 = \varepsilon^{-2}(x_1 + 1)$, $\xi_2 = \varepsilon^{-2}x_2$ the equation of the ellipse will be written in the form $\xi_2^2 = p_0^2\xi_1(1 - {}^1/_2\varepsilon^2\xi_1)$. Passing formally to $\varepsilon = 0$, we obtain the required parabola, and this implies that by handling the solution in exactly the same way we obtain a formula for z^0 with an unknown multiplier which will be determined by comparison with the asymptotic expression (2.5) (here it is convenient to assume that $\xi_2 = \xi_3 = 0$ and $\xi_1 \to -\infty$).

The correctness of the transformations used can be confirmed by substituting the solutions into relations (2.4). Finally, the expression /10/ for the SIC at the edge of external elliptical crack will enable us to calculate the SIC on $\partial\Pi$. The latter depends on the variable $s = \varepsilon^2\xi_1$ in the following manner:

$$\varepsilon^{-1/2}(\pi p_0)^{-1/2}\mu(1-\nu)^{-1}\gamma(0)(s + {}^1/_4 p_0^2\varepsilon^2)^{-1/4} \tag{2.6}$$

By virtue of (2.3) formulas (2.6) and (1.12) are consistent and can be matched.

According to (2.6) the SIC attains its maximum value at the point $s = 0$ or $s = l$ (this fact is mentioned in /10/). If we assume that the bridge Γ_ε was formed as the result of partial fracturing of a larger bridge, then it is natural to assume that the SIC is constant on the resulting edge. The relation (1.2) shows that in this case it is necessary that the function h, i.e. the reduced width of the bridge, does not vanish when $s = 0$ and $s = l$. In other words, we have case 4° when the bridge has blunt ends.

The form of the end zone can be conveniently described in the coordinates $\xi_1 = \varepsilon^{-1}s$, $\xi_2 = \varepsilon^{-1}n$, $\xi_3 = \varepsilon^{-1}z$. Let now Π be a region in a plane which resembles, outside the circle of large radius, a half-strip $\{(\xi_1, \xi_2): \xi_1 > 0, -h_-(0) < \xi_2 < h_+(0)\}$. The problem of the boundary layer then reduces to determining a function z^0 harmonic in $\mathbf{R}^3$ and satisfying relations (2.4) and (2.5). It can be confirmed (using for example the Kelvin transformation and asymptotic formulas from Sect.2°), that the corresponding SIC will tend to a constant value as $\xi_1 \to +\infty$. The problem arises here of determining the contour $\partial\Pi$ for which the solution z^0 of the problem will produce a constant SIC along $\partial\Pi$. We are unable to state whether such a contour exists.

3. Variational inequality for a bridge in equilibrium in the limit. Let a system of cracks form a thin bridge near the contour Γ, of width $\varepsilon H(s)$. We shall assume that changing the load resulted in partial fracturing of the bridge, while the configuration of the cracks outside the neighbourhood of Γ remained as before (according to (1.12) the values of the SIC are large at the bridge edges, and this assumption is credible). We shall interpret the set of bridges resulting from the fracturing as a single bridge Γ_ε of width $\varepsilon h(s)$, which may become equal to zero on some segments, and write $\Upsilon = \{s \in [0, l] : h(s) = 0\}$. The conditions that the crack edges do not close means that

$$0 \leqslant h(s) \leqslant H(s) \quad (s \in [0, l]) \tag{3.1}$$

Let γ be the distribution density of the normal force on the arc $\Gamma \setminus \Upsilon$, appearing in Sect.1 and 2. Let us supplement this density by a zero on Υ and assume, in the course of deriving the variational inequality, that γ is a smooth function on Γ. Then the principal term of the asymptotic expansion will, as before, have the form (1.2). In the case of $h(s) = \gamma(s) = 0$ we have, according to (1.5), $v^0(x) = (J\gamma)(s) - g(s) + O(r)$, $r \to 0$. Since a crack has been formed on Υ, it follows that $v^0 \geqslant 0$ near Υ and

$$h(s) = \gamma(s) = 0 \Rightarrow (J\gamma)(s) - g(s) \geqslant 0 \tag{3.2}$$

A boundary layer appears near the bridge $(\Gamma \setminus \Upsilon)_\varepsilon$, enabling us to calculate an approximation to the SIC, similar to (1.12). We shall assume that the edges $\Gamma_\varepsilon^\pm$ are under the conditions of limit equilibrium. Denoting by K_c the critical value of the SIC we obtain, from (1.12),

$$\gamma(s) = (2\mu)^{-1} K_c (1-\nu)[2\pi\varepsilon h(s)]^{1/2} \equiv \varkappa^{-1/2} h(s)^{1/2} \tag{3.3}$$

This yields the quantity $h(s)$. We note that when $s \in \Gamma \setminus \Upsilon$, Eq.(1.8) holds and we arrive at the formulas

$$\gamma(s) > 0 \Rightarrow \{\pi^{-1}[\ln(\varepsilon k(s)) - 5\ln 2 + \ln(\varkappa\gamma(s)^2)] - b(s)\}\gamma(s) + (J\gamma)(s) - g(s) = 0 \tag{3.4}$$

As usual /11/, relations (3.2) and (3.4) can be written in the form of a variational inequality. By virtue of (3.2) and (3.4) we have

$$(\gamma B(\varkappa\gamma^2),\ \gamma) + (J\gamma,\ \gamma) - (g,\ \gamma) = 0 \tag{3.5}$$

Here $(\,,)$ is a scalar product in $L_2(\Gamma)$ (or its expansion) and $B(\varkappa\gamma^2)$ represents the expression within the braces in (3.4). Formulas (3.2) and (3.4) yield, for any smooth non-negative function β on Γ, the relations

$$(\gamma B(\varkappa\gamma^2),\ \beta) + (J\gamma,\ \beta) - (g,\ \beta) \geqslant 0 \tag{3.6}$$

Subtracting (3.5) from (3.6) we obtain the required variational inequality

$$(\gamma B(\varkappa\gamma^2),\ \gamma-\beta) + (J\gamma,\ \gamma-\beta) \leqslant (g,\ \gamma-\beta) \quad \forall\beta \geqslant 0 \tag{3.7}$$

We can confirm directly that the variational inequality (3.7) is, in fact, the problem of a minimum of the functional

$$\tfrac{1}{2}(\gamma B(\varkappa\gamma^2),\ \gamma) + \tfrac{1}{2}(J\gamma,\ \gamma) - (g + (2\pi)^{-1}\gamma,\ \gamma) \tag{3.8}$$

Let us determine the convex set on which the functional (3.8) is minimized. In accordance with (1.9) we denote by $\mathbf{H}_{\ln}(\Gamma)$ the space of functions on Γ with the norm

$$\left(\int_\Gamma\int_\Gamma |\gamma(s) - \gamma(\tau)|^2 |s-\tau|^{-1} ds\, d\tau + \int_\Gamma |\gamma(s)|^2 ds\right)^{1/2}$$

It was established in /12/ that $\mathbf{H}_{\ln}(\Gamma)$ is a Hermander space /13/ generated by the weight function $\mu(\xi) = (1 + \ln|\xi| + |\ln|\xi||)^{1/2}$. In other words, the norm in $\mathbf{H}_{\ln}(\Gamma)$ is constructed by partitioning the unit from the norms in $\mathbf{H}_{\ln}(\mathbf{R})$ calculated with the help of the Fourier transformation $F_{s\to\xi}$ according to the formula

$$\|\gamma; \mathbf{H}_{\ln}(\Gamma)\| = \int_\mathbf{R} [1 + \ln|\xi| + |\ln|\xi||]\,|(F_{s\to\xi}\gamma)(\xi)|^2 d\xi$$

We stress that the inclusion $\mathbf{H}_{\ln}(\Gamma) \subset L_2(\Gamma)$ is compact (this property leads to a discrete form of the spectrum of integral operator (1.10) used in Sect.1 and 2). We denote by $L_{2,\ln}(\Gamma)$ the linear manifold of functions on Γ such, that $\gamma\mu(\gamma) \in L_2(\Gamma)$. This is the Orlicz space constructed on N-functions $x \mapsto M(x) = x^2\mu(x)^2$ (see e.g. /14/).

Thus the convex set required should be constructed from non-negative functions contained within the intersection $\mathbf{H}_{\ln}(\Gamma) \cap L_{2,\ln}(\Gamma)$.

If we have a separation near Γ_ε (it is necessary to make an assumption which would eliminate the possibility of a contact between the crack edges), then g will be a non-positive function on Γ. Then $\gamma \equiv 0$ will be a solution of the inequality (3.7) corresponding to the case of complete fracturing of the bridge. Such a (complete) fracturing corresponds fully to the idea of the unstable growth of cracks: the SIC increases as $h(s)$ decreases. Nevertheless we cannot make any assertions concerning the uniqueness of the solution γ (even through non-trivial solutions of the inequality (3.7) may exist by virtue of what was said after formula (1.10)). Thus cross-connections between the cracks in the limit equilibrium are possible, although their state should be described as unstable.

4. Variational inequality for a partially fracturing bridge. Since the thinning of the bridge is a locally unstable process, it is convenient to assume that after partial fracturing its width $\varepsilon h(s)$ can be zero or $\varepsilon H(s)$. At the points s at which $h(s) = H(s)$, the condition $\gamma(s) \leqslant \varkappa^{-1/2} H(s)$ of equilibrium of the crack should hold (see (3.3) and (1.12)). We shall therefore consider the problem of minimizing the functional $^1/_2(\gamma B(H), \gamma) + {}^1/_2(J\gamma, \gamma) - (g, \gamma)$ on the convex set $\mathbf{M} = \{\gamma \in \mathbf{H}_{\ln}(\Gamma)\colon\ 0 \leqslant \gamma \leqslant \varkappa^{-1/2} H^{1/2}\}$, or the corresponding variational inequality

$$(\gamma B(H),\ \gamma - \beta) + (J\gamma,\ \gamma - \beta) \leqslant (g,\ \gamma - \beta)\ \ \forall \beta \in \mathbf{M} \tag{4.1}$$

When $g \leqslant 0$ on Γ (the bridge is under the condition of stretching), then trivial solution $\gamma \equiv 0$ exists, but as before, the existence of other, non-trivial solutions cannot be excluded. Let γ be such a smooth solution. Using familiar arguments /11/ we establish the formulas

$$\gamma(s) = 0 \Rightarrow (J\gamma)(s) - g(s) \geqslant 0 \tag{4.2}$$

$$0 < \gamma(s) < \varkappa^{-1/2} H(s)^{1/2} \Rightarrow B(H;\ s)\,\gamma(s) + (J\gamma)(s) - g(s) = 0 \tag{4.3}$$

$$\gamma(s) = \varkappa^{-1/2} H(s)^{1/2} \Rightarrow B(H;\ s)\,\gamma(s) + (J\gamma)(s) - g(s) \leqslant 0 \tag{4.4}$$

We assume that for the solution discussed the inequality sign holds in (4.4) on the set Γ^* of zero measure (only at certain points). Then the necessary relation (1.8) (the equation of equilibrium) will hold outside Γ^* only if $\gamma(s) > 0$ (the bridge remains unbroken at the point s). The conditions $\gamma(s) = 0$ and $(J\gamma)(s) > g(s)$ mean that the crack edges have separated by a positive distance, and there is no bridge at this point any longer. In the case $\gamma(s) = 0$ and $(J\gamma)(s) = g(s)$ both situations are possible (and for this reason the bridge may remain whole on the set $\Gamma^\circ = \{s \in \Gamma\colon\ g(s) = 0\}$ even when $\gamma \equiv 0$). Therefore in the course of solving the variational inequality (4.1) we determine a zone of obligatory fracturing, and a zone in which the bridge is certainly preserved. At the arc Γ° itself the deformations within the body are such, that it becomes immaterial whether a bridge is, or is not there.

We stress that the variational inequality derived in /15/ which describes the variation in the form of a single crack growing quasistatically, differs significantly from (3.7) and (4.1). The problem is, that the paper by Nazarov /15/ is concerned with the study of the stable development of the crack; in particular, a small change in the load led, generally speaking, to a small increase in the free surface, provided that avalanche-type growth of the crack did not occur. In the problem discussed here the bridge is narrow, and the increase in the free surface remains insignificant even in the case when fracturing took place near a large segment of the contour Γ. Quasistatic fracturing of the bridge is not often realized, and the most probable cases are those in which neglecting the inertial terms leads to serious errors. It is precisely these conditions that must be related to the resulting non-uniqueness of the solution and the indefiniteness of the fracturing on the segment $\Gamma^\circ \subset \Gamma$.

Let us finally consider a specific problem. Let two half-spaces be connected along the strip $\Gamma_\varepsilon = \{y \in \mathbf{R}^2\colon |y_2| < \varepsilon\}$, and loaded with a pair of normal concentrated cleaving forces of strenght q°, applied at the points $y = (0, \pm 1)$. The limit contour Γ is the straight line $\{y\colon y_2 = 0\}$. The function g is given by the equation $g(s) = q^\circ(\pi\mu)^{-1}(\nu - 1)(s^2 + 1)^{-1/2}$ where $s = y_1$. The solution of the limit problem is given by formula (1.2), but since the integral over Γ diverges at infinity, the second term in (1.2) has to be regularized anew.

Let us assume that the fracturing took place symmetrically about the y_2 axis, and denote by a the half-length of the fractured part of the bridge. Since $\gamma(s) = 0$ when $|s| < a$ and $\gamma(s) = \gamma(-s)$ when $|s| > a$, we have

$$\int_\Gamma \gamma(\tau)[(s-\tau)^2 + y_2^2]^{-1/2}\,d\tau = \int_a^\infty \gamma(\tau)\{[(s-\tau)^2 + y_2^2]^{-1/2} + [(s+\tau)^2 + y_2^2]^{-1/2}\}\,d\tau =$$

$$\gamma(s)\int_a^{s+a} [(s-\tau)^2 + y_2^2]^{-1/2}\,d\tau + 2\pi(L\gamma)(s) + O(|y_2|) =$$

$$\gamma(s)[\ln(a - s + [(s-a)^2 + y_2^2]^{1/2}) - \ln 2a] + 2\pi(L\gamma)(s) + O(|y_2|)$$

$$(L\gamma)(s) = \int_a^\infty \frac{\gamma(\tau)}{|s+\tau|}\,d\tau + \int_{s+a}^\infty \frac{\gamma(\tau)}{|s-\tau|}\,d\tau + \int_a^{s+a} \frac{\gamma(\tau) - \gamma(s)}{|s-\tau|}\,d\tau$$

According to what was said in Sect.2 and 4, the following relations must hold:

$$\gamma(s)\{\ln(\varepsilon/4) - {}^1/_2\ln(a(s-a))\} + \pi(L\gamma)(s) - \pi g(s) = 0,\quad s > a \tag{4.5}$$

$$\gamma(s) \leqslant \varkappa^{-1/2},\quad s > a \tag{4.6}$$

$$(L\gamma)(s) - g(s) \geqslant 0,\quad s < a \tag{4.7}$$

As in (1.13), the principal term of the asymptotic expansion (in inverse powers of $|\ln\varepsilon|$) is the function $\gamma_0(s) = -\pi|\ln(\varepsilon/4)|^{-1}g$. We note that $\gamma_0(s) > 0$ when $s \geqslant a$. The condition (4.6) holds in this case only if

$$a \geqslant [2(\pi\varepsilon)^{-1}|\ln(\varepsilon/4)|^{-2}(q^\circ K_c^{-1})^2 - 1]^{1/2} \equiv a^\circ(q^\circ) \tag{4.8}$$

Since the function γ_0 is small and g is negative on $[-a, a]$, it follows that condition (4.7) holds.

Analylsis of Eq.(4.5) shows that the function γ possessing minimum smoothness on $[a, \infty)$ should vanish when $s = a$ (thanks to the presence of the increasing terms $-{}^1/_2\gamma(s)\ln(s-a)$ and the unboundedness of the remaining terms in (4.5)). The function γ_0 does not have this property, i.e. an additional boundary layer appears near the point $s = a$ of width $O(\varepsilon^2)$ (the square bracket in (4.5) is equal to zero when $s = a + (16a)^{-1}\varepsilon^2$). Since the asymptotic form is sought from the very beginning with an accuracy of only $O(\varepsilon)$, it follows that we can neglect this phenomenon.

Restricting ourselves in the first approximation γ_0 to the density γ, we shall consider the inequality (4.8) which is necessary for (4.6) to hold. In the case of $q^\circ < q_c \equiv (\pi\varepsilon/2)^{1/2}K_c|\ln(\varepsilon/4)|$ the right-hand side of (4.8) is not defined and the bridge is not subjected to fracturing. If on the other hand the load is greater than the critical value q_c, then the quantity a will have to be positive (a part of the bridge has fractured), but the choice of a will remain arbitrary. The range $[a^\circ(q^\circ), +\infty)$ of possible values of a becomes narrower as the load q° increases. The resulting non-uniqueness can be removed by introducing an additional condition, for example by specifying the opening of the crack found at the point $y = 0$.

Let the mechanism of fracturing of the bridge be quasistatic. The crack stops growing as soon as the SIC ceases to take supercritical values. Therefore, the non-emptiness of the set $\Gamma_c = \{s \in \Gamma: \gamma(s) = \varkappa^{-1/2}H(s)^{1/2}\}$ can serve as a criterion for selecting the "quasistatic" solutions of the variational inequality (4.1). In the present case this condition can hold only in the case $a = a^0(q^0)$.

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